International Journal of Theoretical Physics, Vol. 42, No. 12, December 2003 (© 2003)

Quantization of Anisotropic Rigid Body

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Received May, 15, 2003

A classical model of the rigid body and a quantum one are presented. The mechanics of the rigid body is investigated within the framework of the Lie group theory. Two different prescriptions for the quantization of the arbitrary (i.e., anisotropic or spherical) rigid body are proved to be equivalent.

KEY WORDS: anisotropic rigid body; classical and quantum systems on Lie groups; Laplace–Beltrami operator.

1. INTRODUCTION

There are two prescriptions for the quantization of the rigid body. The first prescription is based on a relation between quantities belonging to the framework of the classical Hamiltonian mechanics and differential operators on the configuration space of a mechanical system. The second one is a correspondence between the kinetic energy expressed by the velocity, not by the momentum (i.e., the quantity belonging to the Lagrangian framework) and the Laplace–Beltrami operator for a certain Riemannian metric on the configuration space of the mechanical system. The two prescriptions are briefly discussed in Section 3.

In many monograph, books and papers within the scope of the mechanics, the problem of the equivalence of the two quantization prescriptions mentioned above is not considered. For example, in Landau and Lifshitz (1965, pp. 383–385) the kinetic energy operator of the rigid body is built in accordance with the first prescription. The second one is omitted.

In Abraham and Marsden (1978, pp. 427–433), the authors develop the first prescription and then define the kinetic energy operator on the grounds of the second prescription, passing over the first one. Afterwards, discussing the quantization of the rigid body, they give the expression for the kinetic energy operator which actually corresponds to the first prescription. There is no explanation of why the two definitions coincide.

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In our paper, filling the gap which exists in the literature, we prove that, for the rigid body, the two quantization prescriptions are equivalent, i.e., they lead to the same quantum Hamiltonian.

Notations. Let V and W be linear spaces. We shall denote by $L(V, W)$ and V^* the linear space of linear mappings $V \to W$ and the dual space of V, respectively. For $p \in V^*$ and $v \in V$, the value of p on v will be denoted by $\langle p, v\rangle$.

Let $\phi \in L(V, W)$. We define the transpose of ϕ, $\phi^* \in L(W^*, V^*)$, by

$$\phi^*(p) = p \circ \phi \text{ for } p \in W^*. \tag{1.1}$$

If ϕ is the isomorphism of V onto W, we can define the contragradient mapping of ϕ, $\tilde{\phi} \in L(V^*, W^*)$, by

$$\tilde{\phi} = (\phi^{-1})^* = (\phi^*)^{-1}. \tag{1.2}$$

Let M and N be differentiable manifolds of class C^∞. By $\mathcal{F}(M)$ (resp. $\chi(M)$) we shall denote the associative algebra of functions $M \to \mathbb{R}$ of class C^∞ (resp. the Lie algebra of vector fields of class C^∞ on M).

Let K be a tensors field on M and $X \in \chi(M)$. We shall denote by $£_X K$ and $X \rfloor$ the Lie derivative of K and the interior product with respect to X, respectively.

Let $f : M \to N$ be a differentiable mapping and $x \in M$. By $(f_*)_x$ (resp. f_*) we shall denote the mapping $T_x M \to T_{f(x)} N$ tangent to f at x (resp. the mapping $TM \to TN$, tangent to f).

2. CLASSICAL MECHANICS OF RIGID BODY

In this section we shall review basic concepts belonging to the classical mechanics of the rigid body. The mechanics is considered within the framework of the Lie group theory.

The configuration space of the rigid body without translational degrees of freedom can be identified with the special orthogonal group $SO(3; \mathbb{R})$. We shall denote $SO(3; \mathbb{R})$ by G.

Let

$$\tau : TG \to G \text{ and } \tau^* : T^*G \to G$$

be the natural projections. The canonical Pfaff form ω on T^*G is a differential one-form on T^*G defined by

$$\omega_p = p \circ ((\tau^*)_*)_p \text{ for } p \in T^*G. \tag{2.1}$$

Given $X \in \chi(G)$ we can uniquely define a Hamiltonian vector field $\bar{X} \in \chi(T^*G)$ as follows:

$$(\tau^*)_* \circ \bar{X} = X \circ (\tau^*), \quad £_{\bar{X}}\omega = 0. \tag{2.2}$$

The function

$$F_X = \langle \omega, \bar{X} \rangle \tag{2.3}$$

is the generator of $\bar{X}$, i.e.,

$$\bar{X} \rfloor \gamma = -dF_X, \quad \gamma = d\omega. \tag{2.4}$$

We shall call $\bar{X}$ the canonical lift of X.

Let e be the identity of G and $\boldsymbol{g}_L$ the Lie algebra of left invariant vector fields on G. We shall identify the Lie algebra $\boldsymbol{g}$ of G with the tangent space T_eG endowed with the Lie bracket induced by $\boldsymbol{g}_L$. For $A \in \boldsymbol{g}$, we shall denote by A^l the unique left invariant vector field on G (i.e. $A^l \in \boldsymbol{g}_L$) such that

$$(A^l)_e = A. \tag{2.5}$$

The angular velocity with respect to the comoving (body-fixed) frame is a mapping

$$\Omega : TG \to \boldsymbol{g}$$

which sends $v \in TG$ into $\Omega(v) \in \boldsymbol{g}$ uniquely determined by the condition

$$((\Omega(v))^l)_g = v, \tag{2.6}$$

where $g = \tau(v)$, i.e., $v \in T_gG$.

Let

$$Ad : G \to GL(\boldsymbol{g})$$

be the adjoint representation of G in $\boldsymbol{g}$. We shall denote by l_g, (resp. r_g) the left (resp. right) translation of G by an element $g \in G$:

We have

$$l_g h = gh, \quad r_g h = hg \text{ for } h \in G.$$

We have

$$\Omega((l_g)_* v) = \Omega(v), \quad \Omega((r_g)_* v) = (Ad_{(g^{-1})})\Omega(v) \text{ for } v \in TG, g \in G. \tag{2.7}$$

The quasimomentum conjugate to Ω is a mapping

$$\sum : T^*G \to \boldsymbol{g}^*$$

given by

$$\sum |T_g^*G = (\Omega|\widetilde{T}_g G) \text{ for } g \in G \tag{2.8}$$

(see (1.1) and (1.2)).

Let $A \in \boldsymbol{g}$ and $F[A]$ be the generator of the canonical lift $\overline{(A^l)} \in \chi(T^*G)$ of $A^l \in \boldsymbol{g}_L \subset \chi(G)$ (see (2.5) and (2.2)). Then Eq. (2.3) becomes

$$F[A] = \left\langle \sum, A \right\rangle, \tag{2.9}$$

i.e.,

$$F[A](p) = \left\langle \sum(p), A \right\rangle, \quad p \in T^*G$$

(see, Sławianowski, 1975a, p. 182).

Let (E_a), $a = 1, 2, 3$, be a basis for $\boldsymbol{g}$ and (E^a) the basis for $\boldsymbol{g}^*$ dual to (E_b), i.e.,

$$\langle E^a, E_b \rangle = \delta^a{}_b. \tag{2.10}$$

We have

$$\Omega = \Omega^a E_a \text{ and } \sum = \sum_a E^a, \tag{2.11}$$

where

$$\Omega^a \in \mathcal{F}(TG), \quad \sum_a \in \mathcal{F}(T^*G).$$

By virtue of (2.10),

$$\sum_a(p) = \left\langle \sum(p), E_a \right\rangle, \quad p \in T^*G. \tag{2.12}$$

Combining Eqs. (2.9) and (2.12) yields

$$\sum_a = F[E_a]. \tag{2.13}$$

Thus $\sum_a$ is the generator of the canonical lift $\overline{(E_a^l)}$ of $E_a^l \in \boldsymbol{g}_L \subset \chi(G)$, where E_a^l and E_a, are related by

$$\left(E_a^l\right)_e = E_a \tag{2.14}$$

(see (2.5)).

The structure constants $C^c{}_{ab}$ of $\boldsymbol{g}$ with respect to the basis (E_a) are given by

$$[E_a, E_b] = C^c{}_{ab} E_c. \tag{2.15}$$

Since $\boldsymbol{g}$ is simple, the structure constants are traceless:

$$C^b{}_{ab} = 0 \tag{2.16}$$

According to notations (2.14) and (2.15), (E_a^l), $a = 1, 2, 3$, is the basis for $\boldsymbol{g}_L$ such that

$$\left[E_a^l, E_b^l\right] = C^c{}_{ab} E_c^l. \tag{2.17}$$

Let B be the Killing–Cartan form of $\boldsymbol{g}$. Since $G = SO(3; \mathbb{R})$ is simple and compact, B is nondegenerate and negative definite. Hence, for any $\alpha > 0$,

$$K_e = -\alpha B \tag{2.18}$$

is the (positive definite) inner product in $\boldsymbol{g} = T_e G$. Using K_e, we define a Riemannian metric K on G by

$$K(v_1, v_2) = K_e(\Omega(v_1), \Omega(v_2)), \quad v_1, v_2 \in T_g G, \qquad g \in G. \tag{2.19}$$

K is called the Killing–Cartan metric on G. It is invariant by both right and left translations of G:

$$K = (r_g)^* K = (l_g)^* K \text{ for all } g \in G \tag{2.20}$$

(see Kobayashi and Nomizu, 1963, p. 155). Hence, both left and right invariant vector fields on G are the Killing fields of the metric K.

The Lagrangian of the rigid body fastened at some point,

$$L : TG \to \mathbb{R},$$

is a function of the form

$$L = T - V \circ \tau, \tag{2.21}$$

where

$$T : TG \to \mathbb{R} \text{ and } V : G \to \mathbb{R}$$

are the kinetic energy and the potential, respectively. The kinetic energy is given by

$$T(v) = \frac{1}{2} I(\Omega(v), \Omega(v)) \text{ for } v \in TG, \tag{2.22}$$

where I is a positive definite inner product in $\boldsymbol{g}$. We shall call I the tensor of inertia with respect to the co-moving (body-fixed) frame. Let us put

$$I_{ab} = I(E_a, E_b). \tag{2.23}$$

Then

$$T(v) = \frac{1}{2} I_{ab} \Omega^a(v) \Omega^b(v), \quad v \in TG, \tag{2.24}$$

where Ω^a are given by first formula (2.11).

Definition 2.1. The rigid body is said to be spherical or isotropic if

$$I = \lambda K_e \tag{2.25}$$

for a certain $\lambda > 0$ (see (2.18)).

Definition 2.2. The rigid body is said to be anisotropic if condition (2.25) is not satisfied.

Let the rigid body be spherical. Substituting (2.25) and (2.19) into formula (2.22) gives

$$T(v) = \frac{1}{2}\lambda K(v, v).$$

Hence, by virtue of (2.20), the kinetic energy T is invariant by both left and right translations of G.

For the anisotropic rigid body, T is invariant by the left translations (see first equation (2.7)), but, in contrast to the isotropic one, T fails to be invariant by the right translations (see second Eq. (2.7)).

The kinetic energy defined as the function on T^*G, i.e.,

$$T : T^*G \to \mathbb{R},$$

have the form

$$T(p) = \frac{1}{2} I^{ab} \sum_a (p) \sum_b (p), \quad p \in T^*G, \tag{2.26}$$

where $[I^{ab}]$ is the inverse matrix of $[I_{ab}]$ and $\sum_a$ are given by second formula (2.11).
The Hamiltonian of the rigid body,

$$H = T + V \circ \tau^* : T^*G \to \mathbb{R},$$

is defined by

$$H = \frac{1}{2} I^{ab} \sum_a \sum_b + V \circ \tau^*. \tag{2.27}$$

3. QUANTUM RIGID BODY

There are two different, standard prescriptions for the quantization of the rigid body. In this section we shall recall them. In the next section we shall prove the prescriptions to be equivalent.

Let $\mathcal{F}^c(G)$ (resp. $\chi^c(G)$) be the associative algebra of functions $G \to \mathbb{C}$ of class C^∞ (resp. the Lie algebra of complex vector fields of class C^∞ on G). The differentiability is understood as the differentiability in the real domain. We shall denote by $End^c(G)$ the associative algebra of $\mathbb{C}$-linear endomorphisms of $\mathcal{F}^c(G)$.

Let $\mathcal{D}^c(G)$ be the associative subalgebra of $End^c(G)$, generated by $\chi^c(G)$ and endomorphisms of the multiplication by the functions belonging to $\mathcal{F}^c(G)$ (i.e., $f(h) = fh$ for $h \in \mathcal{F}^c(G)$ and $f \in \mathcal{F}^c(G) \subset End^c(G)$). We shall call $\mathcal{D} \in \mathcal{D}^c(G)$ a differential operator (of class C^∞) on G.

Let μ be the Haar measure on the compact Lie group $G = SO(3; \mathbb{R})$ and $L^2(G, \mu)$ the Hilbert space of complex functions on G, square-integrable with respect to the measure μ. Obviously,

$$\mathcal{F}^c(G) \subset L^2(G, \mu). \tag{3.1}$$

There exists the unique differential three-form η on G (which does not vanish anywhere) such that for any $f \in L^2(G, \mu)$,

$$\int_G f(g) d\mu(g) = \int_G f\eta. \tag{3.2}$$

Since μ is invariant by right and left translations of G,

$$\pounds_X \eta = 0 \quad \text{for all} \quad X \in \boldsymbol{g}_L. \tag{3.3}$$

According to the general quantization procedure (formulated, e.g., in Mackey, 1963 or Sławianowski, 1975b), the space of the physical states of the quantum rigid body is identified with $\mathcal{F}^c(G) \cap L^2(G, \mu) = \mathcal{F}^c(G)$ (see (3.1)).

Let

$$\varphi \in \mathcal{F}(G).$$

Then

$$\varphi \circ \tau^* : T^*G \to \mathbb{R}.$$

Passing from the classical level to the quantum one, we retain φ (more precisely $\varphi \circ \tau^*$) actually unchanged. Namely, we replace $\varphi \circ \tau^*$ with the operator $\hat{\varphi} \in \mathcal{D}^c(G)$ given by

$$\hat{\varphi}(\psi) = \varphi\psi \text{ for } \psi \in \mathcal{F}^c(G).$$

Hence, we can identify $\hat{\varphi}$ with φ and write

$$\hat{\varphi} = \varphi. \tag{3.4}$$

Let

$$F_X : T^*G \to \mathbb{R}$$

be the generator of the canonical lift $\overline{X}$ of $X \in \chi(G)$ (see (2.2) and (2.3)). On the quantum level, the function F_X is replaced with the operator $\hat{F}_X \in \mathcal{D}^c(G)$ defined as follows:

$$\hat{F}_X = \frac{\hbar}{i} X + \frac{\hbar}{2i} \rho_X, \tag{3.5}$$

where $\hbar = h/2\pi$, h is the Planck constant and $\rho_X \in \mathcal{F}(G) \subset \mathcal{D}^c(G)$ is uniquely determined by the condition

$$\pounds_X \eta = \rho_X \eta$$

(see (3.2)). Hence, by virtue of (3.3),

$$\hat{F}_X = \frac{\hbar}{i} X \text{ for all } X \in \boldsymbol{g}_L. \tag{3.6}$$

Combining Eqs. (2.13) and (3.6) leads to

$$\widehat{\sum}_a = \frac{\hbar}{i} E_a^l. \tag{3.7}$$

We shall now employ the quantization rule presented above to construct a quantum Hamiltonian of the rigid body:

$$\hat{H} \in \mathcal{D}^c(G).$$

Replacing in equation (2.27) the functions $\sum_a$, with the operators $\widehat{\sum}_a$, given by (3.7) and retaining V unchanged (see (3.4)) yields

$$\hat{H} = -\frac{\hbar^2}{2} I^{ab} E_a^l E_b^l = V. \tag{3.8}$$

The quantization prescription discussed above is based on the correspondence between quantities belonging to the framework of the classical Hamiltonian mechanics (i.e., functions $T^*G \to \mathbb{R}$) and differential operators on G. There is another prescription for the quantization of the rigid body. It is a correspondence between the kinetic energy T expressed by the velocity v, not by the momentum p (i.e., the quantity belonging to the Lagrangian framework) and some differential operator on G, defined as follows.

Since the tensor of inertia I is the inner product in $\boldsymbol{g} = T_e G$, it induces a Riemannian metric Γ on G:

$$\Gamma(v_1, v_2) = I(\Omega(v_1), \Omega(v_2)), \quad v_1, v_2 \in T_g G, \ g \in G. \tag{3.9}$$

From first formula (2.7) it follows that Γ is invariant by the left translations of G. This means that the right invariant vector fields on G are the Killing fields of Γ. In contrast to the Killing–Cartan metric K, Γ fails to be invariant by the right translations if the rigid body is anisotropic. Hence, for the anisotropic rigid body the left invariant vector fields on G fail to be the Killing fields of Γ.

Substituting (3.9) into Eq. (2.22) leads to the following formula for the kinetic energy of the rigid body:

$$T(v) = \frac{1}{2}\Gamma(v, v) \quad v \in TG. \tag{3.10}$$

We postulate that, on the quantum level, the function T (given by (3.10) is replaced with the operator of the kinetic energy, $\hat{T} \in \mathcal{D}^c(G)$, defined by

$$\hat{T} = -\frac{\hbar^2}{2} \Delta_\Gamma, \tag{3.11}$$

where Δ_Γ denotes the Laplace–Beltrami operator on G corresponding to the metric Γ. Then the classical formula

$$H = T + V \circ \tau^* : T^*G \to \mathbb{R}$$

is replaced with the quantum one:

$$\hat{H}_\Gamma = \hat{T} + \hat{V} \in \mathcal{D}^c(G).$$

As in the previous case, $\hat{V}$ is given by (3.4). Hence

$$\hat{H}_\Gamma = -\frac{\hbar^2}{2}\Delta_\Gamma + V. \tag{3.12}$$

If the rigid body is spherical,

$$\Gamma = \lambda K$$

for a certain $\lambda > 0$ (see (2.25)). Then it is easy to verify that

$$\widehat{H_\Gamma} = \hat{H},$$

where $\hat{H}$ is defined by (3.8).

In the next section we shall prove that the quantization procedure leading to (3.8) and that leading to (3.12) are also equivalent (i.e., equation (3.13) is also satisfied) if the rigid body is anisotropic.

Remark 3.1. For example, in Abraham and Marsden (1978, p. 433) one can find the equation similar to (3.13). However, the authors do not explain why the operator $\widehat{H_\Gamma}$ coincides with $\hat{H}$.

4. EQUIVALENCE OF TWO PRESCRIPTIONS FOR QUANTIZATION OF ANISOTROPIC RIGID BODY

To begin with, let us recall the concept of the Laplace–Beltrami operator on the Riemannian manifold.

Let M be an n-dimensional Riemannian manifold of class C^∞ with a metric h. We shall denote by ∇ the covariant differentiation induced by the Levi–Civita connection for the metric h.

The Laplace–Beltrami operator on M corresponding to h, $\Delta h \in \mathcal{D}^c(M)$, is defined as follows. Let (U, κ), $\kappa = (x^i)$, $i = 1, \ldots, n$, be any chart of M. We put

$$h|U = h_{ij}dx^i \otimes dx^i, \quad \sqrt{|h|} = \sqrt{|\det[h_{ij}]|}. \tag{4.1}$$

By $[h^{ij}]$ and $\Gamma^i{}_{jk}$ we shall denote the inverse matrix of $[h_{ij}]$ and the components of the Levi–Civita connection for h, respectively. The value of Δ_h on $f \in \mathcal{F}^c(M)$ is defined by

$$(\Delta_h f)|U = h^{ij}\nabla_i(\Delta_j f). \tag{4.2}$$

Using the identities

$$\nabla h = 0 \text{ and } \Gamma^i{}_{jk} = \Gamma^i{}_{kj}$$

turns Eq. (4.2) into the form

$$(\Delta_h f)|U = \frac{1}{\sqrt{|h|}} \left(h^{ij} \sqrt{|h|} f_{,j} \right)_{,i}. \tag{4.3}$$

Obviously, the commas in (4.3) denote partial differentiation with respect to the coordinates of (U, κ).

Let $\mathcal{U}$ be an open set of M and $X_1, \ldots, X_n, \in \chi(\mathcal{U})$ such that (X_A), $A = 1, \ldots, n$, is the field of linear frames. By (ω^A), $A = 1, \ldots, n$, we shall denote the field of linear coframes dual to (X_A), *i.e.*, $\langle \omega^A, X_B \rangle = \delta^A_B$.

We define a system of functions $\gamma^A{}_{BC} \in \mathcal{F}(\mathcal{U})$ by

$$[X_B, X_C] = \gamma^A{}_{BC} X_A. \tag{4.4}$$

we call $\gamma^A{}_{BC}$ the structural functions of (X_A). We put

$$h|\mathcal{U} = h_{AB}\omega^A \otimes \omega^B, \quad \sqrt{|h^X|} = \sqrt{|\det[h_{AB}]|}. \tag{4.5}$$

By $[h^{AB}]$ we shall denote the inverse matrix of $[h_{AB}]$.

Let (U, κ), $\kappa = (x^i)$ be any chart of M such that $U \cap \mathcal{U} \neq \emptyset$. We set

$$X_A|U \cap \mathcal{U} = X^i_A \frac{\partial}{\partial x^i}, \quad \omega^A|\mathcal{U} \cap U = \omega^A_i dx^i. \tag{4.6}$$

Combining Eqs. (4.4) and (4.6) leads to

$$X^i_A \left(\omega^A{}_{j,k} - \omega^A{}_{k,j} \right) = \gamma^A{}_{BC} X^i_A \omega^B_j \omega^C_\kappa. \tag{4.7}$$

Replacing on the right-hand side of Eq. (4.3) $\frac{\partial}{\partial x^i}$, $\sqrt{|h|}$ and h^{ij} with X_A, $\sqrt{|h^X|}$, and h^{AB}, respectively, we obtain the value of some differential operator on the function f. In other words, we define $\Delta_X \in \mathcal{D}^c(\mathcal{U})$ by

$$\Delta_X f = \frac{1}{\sqrt{|h^X|}} X_A \left(h^{AB} \sqrt{|h^X|} X_B f \right) \tag{4.8}$$

for $f \in \mathcal{F}^c(\mathcal{U})$.

Theorem 4.1. *A necessary and sufficient condition that*

$$\Delta_X = \Delta_h|\mathcal{U} \tag{4.9}$$

is that the structural functions $\gamma^A{}_{BC}$ be traceless, i.e.,

$$\gamma^A{}_{BA} = 0 \quad B = 1, \ldots, n. \tag{4.10}$$

Proof: Let (U, κ), $\kappa = (x^i)$ be a chart of M such that $U \cap \mathcal{U} \neq \emptyset$. By virtue of Eqs. (4.1), (4.5), and (4.6),

$$h^{AB} = \omega_i^A \omega_j^B h^{ij}, \quad \sqrt{|h^X|} = |\det[X_C^k]_{n\times n}|\sqrt{|h|}. \tag{4.11}$$

Substituting formulae (4.11) and first equation (4.6) into (4.8), and then using (4.3) yields

$$(\Delta_X f)|U \cap \mathcal{U} = (\Delta_h f)|U \cap \mathcal{U} + R^i f_{,i},$$
$$R^i = \left(\det\left[X_B^l\right]\right)^{-1} X_A^j \left(\omega_k^A \det\left[X_B^l\right]\right)_{,l} h^{ki}.$$

Hence, Eq. (4.9) is satisfied if and only if

$$R^i f_{,i} = 0 \text{ for all } f \in \mathcal{F}^c(\mathcal{U}). \tag{4.12}$$

Since R^i do not depend on f, condition (4.12) is equivalent to the following:

$$R^i = 0, \quad i = 1, , \ldots, n. \tag{4.13}$$

By a straightforward calculation, we turn the system of Eqs. (4.13) into the form

$$X_A^i({\omega^A}_{j,i} - {\omega^A}_{i,j}) = 0, \quad j = 1, \ldots, n. \tag{4.14}$$

By virtue of (4.7), system (4.14) is equivalent to (4.10). Hence, we showed that Eq. (4.9) is satisfied if and only if system (4.10) is satisfied. □

We are now in a position to prove that the two quantization prescriptions presented in Section 3 are equivalent.

Theorem 4.2. *Let $\hat{H}$ (resp. $\hat{H}_\Gamma$) be the quantum Hamiltonian of the arbitrary (i.e., anisotropic or spherical) rigid body defined by (3.8) (resp. (3.12)). Then*

$$\hat{H} = \hat{H}_\Gamma. \tag{4.15}$$

Proof: Let us express the Laplace–Beltrami operator Δ_Γ (corresponding to the left invariant metric Γ on G) in terms of a distinguished field of linear frames on G, namely the basis (E_a^l), $a = 1, 2, 3$, for the Lie algebra $\boldsymbol{g}_\Gamma$, of left invariant vector fields on G.

Let (β^a), $a = 1, 2, 3$, be the field of linear coframes on G dual to (E_a^l). We put

$$\Gamma = \Gamma_{ab}\beta^a \otimes \beta^b.$$

Then $\Gamma_{ab} \in \mathcal{F}(G)$ are given by

$$\Gamma_{ab} = \Gamma(E_a^l, E_b^l). \tag{4.16}$$

Let

$$\sqrt{|\Gamma|} = \sqrt{|\det[\Gamma_{ab}]|}$$

and $[\Gamma^{ab}]$ be the inverse matrix of $[\Gamma_{ab}]$.

From (2.17) it follows that the structural functions of the field of linear frames (E^l_a) (defined by (4.4)) are simply the structure constants $C^c{}_{ab}$ of $\boldsymbol{g}_L$. By virtue of (2.16), condition (4.10) is satisfied. Then Theorem 4.1 implies that

$$\Delta_\Gamma f = \frac{1}{\sqrt{|\Gamma|}} E^l_a \left(\Gamma^{ab} \sqrt{|\Gamma|} E^l_b f\right) \tag{4.17}$$

for $f \in \mathcal{F}^c(G)$.

Substituting (3.9) into (4.16) leads to

$$\Gamma_{ab}(g) = I\left(\Omega\left(\left(E^l_a\right)_g\right), \Omega\left(\left(E^l_b\right)_g\right)\right), \quad g \in G.$$

Combining Eq. (2.6) and (2.14) gives

$$\Omega((E^l_a)_g) = E_a, \quad g \in G.$$

Hence, by virtue of (2.23),

$$\Gamma_{ab}(g) = I_{ab} \quad \text{for} \quad g \in G. \tag{4.18}$$

The components I_{ab} of the tensor of inertia are obviously constants. Thus equation (4.18) implies that Γ_{ab} are the constant functions on G. Then $\sqrt{|\Gamma|}$ and Γ^{ab} are the constant functions, too. Hence formula (4.17) can be written as

$$\Delta_\Gamma f = \Gamma^{ab} E^l_a \left(E^l_b f\right).$$

By virtue of (4.18),

$$\Gamma^{ab} = I^{ab},$$

where $[I^{ab}]$ is the inverse matrix of $[I_{ab}]$. Thus

$$\Delta_\Gamma f = I^{ab} E^l_a \left(E^l_b f\right) \text{ for } f \in \mathcal{F}^c(G).$$

Hence

$$\Delta_\Gamma = I^{ab} E^l_a E^l_b. \tag{4.19}$$

Substituting (4.19) into formula (3.12) yields

$$\hat{H}_\Gamma = -\frac{\hbar}{2} I^{ab} E^l_a E^l_b + V. \tag{4.20}$$

Comparing (4.20) with (3.8), we find that Eq. (4.15) is satisfied. □

ACKNOWLEDGMENT

I thank Professor J. J. Sławianowski for many helpful discussions.

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